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Invariants of triangular Lie algebras

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Abstract

Triangular Lie algebras are the Lie algebras which can be faithfully represented by triangular matrices of any finite size over the real/complex number field. In the paper invariants ('generalized Casimir operators') are found for three classes of Lie algebras, namely those which are either strictly or non-strictly triangular, and for so-called special upper triangular Lie algebras. Algebraic algorithm of Boyko *et al* (2006 *J. Phys. A: Math. Gen.* **39** 5749 (*Preprint* math-ph/0602046)), developed further in Boyko *et al* (2007 *J. Phys. A: Math. Theor.* **40** 113 (*Preprint* math-ph/0606045)), is used to determine the invariants. A conjecture of Tremblay and Winternitz (2001 *J. Phys. A: Math. Gen.* **34** 9085), concerning the number of independent invariants and their form, is corroborated.

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1. Introduction

The invariants of Lie algebras are one of their defining characteristics. They have numerous applications in different fields of mathematics and physics, in which Lie algebras arise (representation theory, integrability of Hamiltonian differential equations, quantum numbers, etc). In particular, the polynomial invariants of a Lie algebra exhaust its set of Casimir operators, i.e., the centre of its universal enveloping algebra. That is why non-polynomial invariants are also called generalized Casimir operators, and the usual Casimir operators are seen as 'trivial' generalized Casimir operators. Since the structure of invariants strongly depends on the structure of the algebra and the classification of all (finite-dimensional) Lie algebras is an inherently difficult problem (actually unsolvable), it seems to be impossible to elaborate a complete theory for generalized Casimir operators in the general case. Moreover, if the classification of a class of Lie algebras is known, then the invariants of such algebras can be described exhaustively. These problems have already been solved for the semi-simple and low-dimensional Lie algebras, and also for the physically relevant Lie algebras of fixed dimensions (see, e.g., references in [3, 7, 8, 18, 19]).

The actual problem is the investigation of generalized Casimir operators for classes of solvable Lie algebras or non-solvable Lie algebras with non-trivial radicals of arbitrary finite dimension. There are a number of papers on the partial classification of such algebras and the subsequent calculation of their invariants [1, 6, 7, 14–16, 20–23]. In particular, Tremblay and Winternitz [22] classified all the solvable Lie algebras with the nilradicals isomorphic to the nilpotent algebra $\mathfrak{t}_0(n)$ of strictly upper triangular matrices for any fixed dimension n. Then in [23] invariants of these algebras were considered. The case $n = 4$ was investigated exhaustively. After calculating the invariants for a sufficiently large value of n, Tremblay and Winternitz made conjectures for an arbitrary n on the number and form of functionally independent invariants of the algebra $\mathfrak{t}_0(n)$, and the 'diagonal' solvable Lie algebras having $\mathfrak{t}_0(n)$ as their nilradicals and possessing either the maximal (equal to $n - 1$) or minimal (one) number of nilindependent elements. A statement on a functional basis of invariants was only proved completely for the algebra $\mathfrak{t}_0(n)$. The infinitesimal invariant criterion was used for the construction of the invariants. Such an approach entails the necessity of solving a system of ρ first-order linear partial differential equations, where ρ has the order of the algebra's dimension. This is why the calculations were very cumbersome and results were obtained due to the thorough mastery of the method.

In this paper, we use our original algebraic method for the construction of the invariants ('generalized Casimir operators') of Lie algebras via the moving frames approach [3, 4]. The algorithm makes use of the knowledge of the associated inner automorphism groups and Cartan's method of moving frames in its Fels–Olver version [9, 10]. (For modern developments about the moving frame method and more references, see also [17].) Unlike standard infinitesimal methods, it allows us to avoid solving systems of differential equations, replacing them instead by algebraic equations. As a result, the application of the algorithm is simpler. Note that a closed approach was earlier proposed in [12, 13, 19] for the specific case of inhomogeneous algebras.

The invariants of three classes of triangular Lie algebras are exhaustively investigated (below n is an arbitrary integer):

- nilpotent Lie algebras $\mathfrak{t}_0(n)$ of $n \times n$ strictly upper triangular matrices (section 3);
- solvable Lie algebras $\mathfrak{t}(n)$ of $n \times n$ upper triangular matrices (section 4);
- solvable Lie algebras $\mathfrak{st}(n)$ of $n \times n$ special upper triangular matrices (section 5).

The triangular algebras are especially interesting due to their 'universality' properties. More precisely, any finite-dimensional nilpotent Lie algebra is isomorphic to a subalgebra of $\mathfrak{t}_0(n)$. Similarly, any finite-dimensional solvable Lie algebra over an algebraically closed field of characteristic 0 (e.g., over $\mathbb{C}$) can be embedded as a subalgebra in $\mathfrak{t}(n)$ (or $\mathfrak{st}(n)$).

We have adapted and optimized our algorithm for the specific case of triangular Lie algebras via special double enumeration of basis elements, individual choice of coordinates in the corresponding inner automorphism groups and an appropriate modification of the normalization procedure of the moving frame method. As a result, the problems related to the construction of functional bases of invariants are reduced for the algebras $\mathfrak{t}_0(n)$ and $\mathfrak{t}(n)$ to solving *linear systems of algebraic equations*! Let us note that due to the natural embedding of $\mathfrak{st}(n)$ to $\mathfrak{t}(n)$ and the representation $\mathfrak{t}(n) = \mathfrak{st}(n) \oplus Z(\mathfrak{t}(n))$, where $Z(\mathfrak{t}(n))$ is the centre of $\mathfrak{t}(n)$, we can construct a basis in the set of invariants of $\mathfrak{st}(n)$ without the usual calculations from a previously found basis in the set of invariants of $\mathfrak{t}(n)$.

We re-prove the statement for a basis of invariants of $\mathfrak{t}_0(n)$, which was first constructed in [23] using the infinitesimal method in a heuristic way, thereafter constructed in [4] using an empiric technique based on the exclusion of parameters within the framework of the algebraic method. The aim of this paper in considering $\mathfrak{t}_0(n)$ is to test and better understand the technique of working with triangular algebras. The calculations for $\mathfrak{t}(n)$ are similar, albeit more complex, although they are much clearer and easier than under the standard infinitesimal approach.

As proved in [22], there is a unique algebra with the nilradical $\mathfrak{t}_0(n)$ that contains a maximum possible number $(n-1)$ of nilindependent elements. A conjecture on the invariants of this algebra is formulated in proposition 1 of [23]. We show that this algebra is isomorphic to $\mathfrak{st}(n)$. As a result, the conjecture by Tremblay and Winternitz on its invariants is effectively proved.

2. The algorithm

The applied algebraic algorithm was first proposed in [3] and then developed in [4]. Ibid it was effectively tested for the low-dimensional Lie algebras and a wide range of solvable Lie algebras with a fixed structure of nilradicals. The presentation of the algorithm here differs from [3, 4], the differences being important within the framework of applications.

For convenience of the reader and to introduce some necessary notations, before the description of the algorithm, we briefly repeat the preliminaries given in [3, 4] about the statement of the problem of calculating Lie algebra invariants, and on the implementation of the moving frame method [9, 10]. The comparative analysis of the standard infinitesimal and the presented algebraic methods, as well as their modifications, is given in the second part of this section.

Consider a Lie algebra $\mathfrak{g}$ of dimension $\dim \mathfrak{g} = n < \infty$ over the complex or real field and the corresponding connected Lie group G. Let $\mathfrak{g}^*$ be the dual space of the vector space $\mathfrak{g}$. The map $\mathrm{Ad}^* : G \to GL(\mathfrak{g}^*)$, defined for any $g \in G$ by the relation

$$\langle \mathrm{Ad}^*_g x, u\rangle = \langle x, \mathrm{Ad}_{g^{-1}} u\rangle \qquad \text{for all} \quad x \in \mathfrak{g}^* \quad \text{and} \quad u \in \mathfrak{g}$$

is called the *coadjoint representation* of the Lie group G. Here $\mathrm{Ad} : G \to GL(\mathfrak{g})$ is the usual adjoint representation of G in $\mathfrak{g}$, and the image Ad_G of G under Ad is the inner automorphism group $\mathrm{Int}(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$. The image of G under Ad^* is a subgroup of $GL(\mathfrak{g}^*)$ and is denoted by Ad^*_G.

A function $F \in C^\infty(\mathfrak{g}^*)$ is called an *invariant* of Ad^*_G if $F(\mathrm{Ad}^*_g x) = F(x)$ for all $g \in G$ and $x \in \mathfrak{g}^*$. The set of invariants of Ad^*_G is denoted by $\mathrm{Inv}(\mathrm{Ad}^*_G)$. The maximal number $N_\mathfrak{g}$ of functionally independent invariants in $\mathrm{Inv}(\mathrm{Ad}^*_G)$ coincides with the codimension of the regular orbits of Ad^*_G, i.e., it is given by the difference

$$N_\mathfrak{g} = \dim \mathfrak{g} - \mathrm{rank}\, \mathrm{Ad}^*_G.$$

Here $\mathrm{rank}\, \mathrm{Ad}^*_G$ denotes the dimension of the regular orbits of Ad^*_G and will be called the *rank of the coadjoint representation* of G (and of $\mathfrak{g}$). It is a basis independent characteristic of the algebra $\mathfrak{g}$, the same as $\dim \mathfrak{g}$ and $N_\mathfrak{g}$.

To calculate the invariants explicitly, one should fix a basis $\mathcal{E} = \{e_1, \ldots, e_n\}$ of the algebra $\mathfrak{g}$. It leads to fixing the dual basis $\mathcal{E}^* = \{e_1^*, \ldots, e_n^*\}$ in the dual space $\mathfrak{g}^*$ and to the identification of $\mathrm{Int}(\mathfrak{g})$ and Ad^*_G with the associated matrix groups. The basis elements $e_1, \ldots, e_n$ satisfy the commutation relations $[e_i, e_j] = \sum_{k=1}^n c_{ij}^k e_k$, $i, j = 1, \ldots, n$, where c_{ij}^k are components of the tensor of structure constants of $\mathfrak{g}$ in the basis $\mathcal{E}$.

Let $x \to \check{x} = (x_1, \ldots, x_n)$ be the coordinates in $\mathfrak{g}^*$ associated with $\mathcal{E}^*$. Given any invariant $F(x_1, \ldots, x_n)$ of Ad^*_G, one finds the corresponding invariant of the Lie algebra $\mathfrak{g}$ by symmetrization, Sym $F(e_1,\ldots,e_n)$, of F. It is often called a *generalized Casimir operator* of $\mathfrak{g}$. If F is a polynomial, Sym $F(e_1,\ldots,e_n)$ is a usual *Casimir operator*, i.e., an element of the centre of the universal enveloping algebra of $\mathfrak{g}$. More precisely, the symmetrization operator Sym acts only on the monomials of the forms $e_{i_1}\cdots e_{i_r}$, where there are non-commuting elements among $e_{i_1},\ldots,e_{i_r}$, and is defined by the formula

$$\mathrm{Sym}\big(e_{i_1}\cdots e_{i_r}\big)=\frac{1}{r!}\sum_{\sigma\in S_r}e_{i_{\sigma_1}}\cdots e_{i_{\sigma_r}},$$

where $i_1,\ldots,i_r$ take values from 1 to n, $r\geqslant 2$. The symbol S_r denotes the permutation group consisting of r elements. The set of invariants of $\mathfrak{g}$ is denoted by $\mathrm{Inv}(\mathfrak{g})$.

A set of functionally independent invariants $F^l(x_1,\ldots,x_n)$, $l=1,\ldots,N_{\mathfrak{g}}$, forms *a functional basis* (*fundamental invariant*) of $\mathrm{Inv}(\mathrm{Ad}^*_G)$, i.e., any invariant $F(x_1,\ldots,x_n)$ can be uniquely represented as a function of $F^l(x_1,\ldots,x_n)$, $l=1,\ldots,N_{\mathfrak{g}}$. Accordingly the set of Sym $F^l(e_1,\ldots,e_n)$, $l=1,\ldots,N_{\mathfrak{g}}$, is called a basis of $\mathrm{Inv}(\mathfrak{g})$.

Our task here is to determine the basis of the functionally independent invariants for Ad^*_G, and then to transform these invariants into the invariants of the algebra $\mathfrak{g}$. Any other invariant of $\mathfrak{g}$ is a function of the independent ones.

Let us recall some facts from [9, 10] and adapt them to the particular case of the coadjoint action of G on $\mathfrak{g}^*$. Let $\mathcal{G}=\mathrm{Ad}^*_G\times\mathfrak{g}^*$ denote the trivial left principal Ad^*_G-bundle over $\mathfrak{g}^*$. The right regularization $\widehat{R}$ of the coadjoint action of G on $\mathfrak{g}^*$ is the diagonal action of Ad^*_G on $\mathcal{G}=\mathrm{Ad}^*_G\times\mathfrak{g}^*$. It is provided by the map $\widehat{R}_g(\mathrm{Ad}^*_h,x)=(\mathrm{Ad}^*_h\cdot\mathrm{Ad}^*_{g^{-1}},\mathrm{Ad}^*_g x)$, $g,h\in G$, $x\in\mathfrak{g}^*$, where the action on the bundle $\mathcal{G}=\mathrm{Ad}^*_G\times\mathfrak{g}^*$ is regular and free. We call $\widehat{R}_g$ the *lifted coadjoint action* of G. It projects back to the coadjoint action on $\mathfrak{g}^*$ via the Ad^*_G-equivariant projection $\pi_{\mathfrak{g}^*}:\mathcal{G}\to\mathfrak{g}^*$. Any *lifted invariant* of Ad^*_G is a (locally defined) smooth function from $\mathcal{G}$ to a manifold, which is invariant with respect to the lifted coadjoint action of G. The function $\mathcal{I}:\mathcal{G}\to\mathfrak{g}^*$ given by $\mathcal{I}=\mathcal{I}(\mathrm{Ad}^*_g,x)=\mathrm{Ad}^*_g x$ is the *fundamental lifted invariant* of Ad^*_G, i.e., $\mathcal{I}$ is a lifted invariant, and any lifted invariant can be locally written as a function of $\mathcal{I}$. Using an arbitrary function $F(x)$ on $\mathfrak{g}^*$, we can produce the lifted invariant $F\circ\mathcal{I}$ of Ad^*_G by replacing x with $\mathcal{I}=\mathrm{Ad}^*_g x$ in the expression for F. Ordinary invariants are particular cases of lifted invariants, where one identifies any invariant formed as its composition with the standard projection $\pi_{\mathfrak{g}^*}$. Therefore, ordinary invariants are particular functional combinations of lifted ones that happen to be independent of the group parameters of Ad^*_G.

The *algebraic algorithm* for finding invariants of the Lie algebra $\mathfrak{g}$ is briefly formulated in the following four steps.

(1) *Construction of the generic matrix $B(\theta)$ of* Ad^*_G. $B(\theta)$ is the matrix of an inner automorphism of the Lie algebra $\mathfrak{g}$ in the given basis $e_1,\ldots,e_n$, $\theta=(\theta_1,\ldots,\theta_r)$ is a complete tuple of group parameters (coordinates) of $\mathrm{Int}(\mathfrak{g})$, and $r=\dim\mathrm{Ad}^*_G=\dim\mathrm{Int}(\mathfrak{g})=n-\dim\mathrm{Z}(\mathfrak{g})$, where $\mathrm{Z}(\mathfrak{g})$ is the centre of $\mathfrak{g}$.

(2) *Representation of the fundamental lifted invariant*. The explicit form of the fundamental lifted invariant $\mathcal{I}=(\mathcal{I}_1,\ldots,\mathcal{I}_n)$ of Ad^*_G in the chosen coordinates $(\theta,\check{x})$ in $\mathrm{Ad}^*_G\times\mathfrak{g}^*$ is $\mathcal{I}=\check{x}\cdot B(\theta)$, i.e., $(\mathcal{I}_1,\ldots,\mathcal{I}_n)=(x_1,\ldots,x_n)\cdot B(\theta_1,\ldots,\theta_r)$.

(3) *Elimination of parameters by normalization*. We choose the maximum possible number ρ of lifted invariants $\mathcal{I}_{j_1},\ldots,\mathcal{I}_{j_\rho}$, constants $c_1,\ldots,c_\rho$ and group parameters $\theta_{k_1},\ldots,\theta_{k_\rho}$ such that the equations $\mathcal{I}_{j_1}=c_1,\ldots,\mathcal{I}_{j_\rho}=c_\rho$ are solvable with respect to $\theta_{k_1},\ldots,\theta_{k_\rho}$. After substituting the found values of $\theta_{k_1},\ldots,\theta_{k_\rho}$ into the other lifted invariants, we obtain $N_{\mathfrak{g}}=n-\rho$ expressions $F^l(x_1,\ldots,x_n)$ without θ's.

(4) *Symmetrization*. The functions $F^l(x_1,\ldots,x_n)$ necessarily form a basis of $\mathrm{Inv}(\mathrm{Ad}^*_G)$. They are symmetrized to Sym $F^l(e_1,\ldots,e_n)$. It is the desired basis of $\mathrm{Inv}(\mathfrak{g})$.

Let us give some remarks on the steps of the algorithm, mainly paying attention to the special features of its variation in this paper, and where it differs from the conventional infinitesimal method.

Usually, the second canonical coordinate on $\mathrm{Int}(\mathfrak{g})$ is enough for the first step, although sometimes, the first canonical coordinate on $\mathrm{Int}(\mathfrak{g})$ is the more appropriate choice. In both the cases, the matrix $B(\theta)$ is calculated by exponentiation from matrices associated with the structure constants. Often the parameters θ are additionally transformed in a trivial manner (signs, renumbering, re-denotation, etc) for simplification of the final presentation of $B(\theta)$. It is also sometimes convenient for us to introduce 'virtual' group parameters corresponding to the centre basis elements. Efficient exploitation of the algorithm imposes certain constrains on the choice of bases for $\mathfrak{g}$, in particular, in the enumeration of their elements; thus automatically yielding simpler expressions for elements of $B(\theta)$ and, therefore, expressions of the lifted invariants. In some cases the simplification is considerable.

In contrast with the general situation, for the triangular Lie algebras we use special coordinates for their inner automorphism groups, which naturally harmonize with the canonical matrix representations of the corresponding Lie groups and with special 'matrix' enumeration of the basis elements. The application of the individual approach results in the clarification and a substantial reduction of all calculations. In particular, algebraic systems solved under normalization become linear with respect to their parameters.

Since $B(\theta)$ is a general form matrix from $\mathrm{Int}(\mathfrak{g})$, it should not be adapted in any way for the second step.

Indeed, the third step of the algorithm can involve different techniques of elimination of parameters which are also based on using an explicit form of lifted invariants [3, 4]. The applied normalization procedure [9, 10] can also be subject to some variations and can be applied in a more involved manner.

As a rule, in complicated cases the main difficulty is created by the determination of the number ρ, who is actually equal to $\mathrm{rank}\,\mathrm{Ad}^*_G$, which is equivalent to finding the maximum number $N_\mathfrak{g}$ of functionally independent invariants in $\mathrm{Inv}(\mathrm{Ad}^*_G)$, since $N_\mathfrak{g} = \dim\mathfrak{g} - \mathrm{rank}\,\mathrm{Ad}^*_G$. The rank ρ of the coadjoint representation Ad^*_G can be calculated in different ways, e.g., by the closed formulae

$$\rho = \max_{\check{x}\in\mathbb{R}^n} \mathrm{rank}\left(\sum_{k=1}^{n} c_{ij}^k x_k\right)_{i,j=1}^{n}, \qquad \rho = \max_{\check{x}\in\mathbb{R}^n}\max_{\theta\in\mathbb{R}^r} \mathrm{rank}\,\frac{\partial\mathcal{I}}{\partial\theta}$$

or with the use of indirect argumentation. The first formula is native to the infinitesimal approach to invariants (see, e.g., [5, 16, 18, 23] and other references) since it gives the number of algebraically independent differential equations in the linear system of first-order partial differential equations $\sum_{j,k=1}^{n} c_{ij}^k x_k F_{x_j} = 0$, which arises under this approach and is the infinitesimal criterion for invariants of the algebra $\mathfrak{g}$ under the fixed basis $\mathcal{E}$. The second formula shows that $\mathrm{rank}\,\mathrm{Ad}^*_G$ coincides with the maximum dimension of a nonsingular submatrix in the Jacobian matrix $\partial\mathcal{I}/\partial\theta$. The tuples of lifted invariants and parameters associated with this submatrix are appropriate for the normalization procedure, where the constants $c_1, \ldots, c_\rho$ are chosen to lie in the range of values of the corresponding lifted invariants.

If ρ is known then the sufficient number ($N_\mathfrak{g} = \dim\mathfrak{g} - \rho$) of functionally independent invariants can be found with various 'empiric' techniques in the frameworks of both the infinitesimal and algebraic approaches. For example, expressions of candidates for invariants can be deduced from invariants of similar low-dimensional Lie algebras and then tested via substitution to the infinitesimal criterion for invariants. It is the method used in [23] to describe invariants of the Lie algebra $\mathfrak{t}_0(n)$ of strictly upper triangular $n \times n$ matrices for any fixed $n \geqslant 2$. In the framework of the algebraic approach, invariants can be constructed via the combination of lifted invariants in expressions not depending on the group parameters [9, 10]. This method was applied, in particular, to low-dimensional algebras and the algebra $\mathfrak{t}_0(n)$ [3, 4]. Other empiric techniques, e.g., based on commutator properties [2] also can be used.

At the same time, a basis of $\mathrm{Inv}(\mathrm{Ad}^*_G)$ may be constructed without first determining the number of basis elements. Since under such consideration the infinitesimal approach leads to the necessity of the complete integration of the partial differential equations from the infinitesimal invariant criterion, the domain of its applicability seems quite narrow (low-dimensional algebras and Lie algebra of special simple structure). A similar variation of the algebraic method is based on the following obvious statement.

Proposition 1. *Let* $\mathcal{I} = (\mathcal{I}_1, \ldots, \mathcal{I}_n)$ *be a fundamental lifted invariant, for the lifted invariants* $\mathcal{I}_{j_1}, \ldots, \mathcal{I}_{j_\rho}$ *and some constants* $c_1, \ldots, c_\rho$ *the system* $\mathcal{I}_{j_1} = c_1, \ldots, \mathcal{I}_{j_\rho} = c_\rho$ *be solvable with respect to the parameters* $\theta_{k_1}, \ldots, \theta_{k_\rho}$ *and substitution of the found values of* $\theta_{k_1}, \ldots, \theta_{k_\rho}$ *into the other lifted invariants result in* $m = n - \rho$ *expressions* $\hat{\mathcal{I}}_l$, $l = 1, \ldots, m$, *depending only on* x*'s. Then* $\rho = \operatorname{rank} \mathrm{Ad}^*_G$, $m = N_{\mathfrak{g}}$ *and* $\hat{\mathcal{I}}_1, \ldots, \hat{\mathcal{I}}_m$ *form a basis of* $\mathrm{Inv}(\mathrm{Ad}^*_G)$.

Our experience on the calculation of invariants of a wide range of Lie algebras shows that the version of the algebraic method, which is based on proposition 1, is most effective. It is the version that is used in this paper.

Note that the normalization procedure is difficult to be made algorithmic. There is a big ambiguity in the choice of the normalization equations. We can take different tuples of ρ lifted invariants and ρ constants, which lead to systems solvable with respect to ρ parameters. Moreover, lifted invariants can be additionally combined before forming a system of normalization equations or substitution of found values of parameters. Another possibility is to use a floating system of normalization equations (see section 6.2 of [4]). This means that elements of an invariant basis are constructed under different normalization constraints. The choice of an optimal method results in a considerable reduction of calculations and a practical form of constructed invariants.

3. Nilpotent algebra of strictly upper triangular matrices

Consider the nilpotent Lie algebra $\mathfrak{t}_0(n)$ isomorphic to the one of the strictly upper triangular $n \times n$ matrices over the field $\mathbb{F}$, where $\mathbb{F}$ is either $\mathbb{C}$ or $\mathbb{R}$. $\mathfrak{t}_0(n)$ has dimension $n(n-1)/2$. It is the Lie algebra of the Lie group $T_0(n)$ of upper unipotent $n \times n$ matrices, i.e., upper triangular matrices with entries equal to 1 in the diagonal.

As mentioned above, the basis of $\mathrm{Inv}(\mathfrak{t}_0(n))$ was first constructed in a heuristic way in [23] within the framework of the infinitesimal approach. This result was re-obtained in [4] with the use of the pure algebraic algorithm first proposed in [3] and developed in [4]. Also, it is the unique example included among the wide variety of solvable Lie algebras investigated in [4], in which the 'empiric' technique of excluding group parameters from lifted invariants was applied. Although this technique was very effective in constructing a set of functionally independent invariants (calculations were reduced via a special representation of the coadjoint action to a trivial identity using matrix determinants, see note 2), the main difficulty was in proving that the set of invariants is a basis of $\mathrm{Inv}(\mathfrak{t}_0(n))$, i.e. cardinality of the set equals the maximum possible number of functionally independent invariants. Under the infinitesimal approach [23] the main difficulty was the same.

In this section, we construct a basis of $\mathrm{Inv}(\mathfrak{t}_0(n))$ with the algebraic algorithm but exclude group parameters from lifted invariants by the normalization procedure. In contrast with the previous expositions (section 3 of [23] and section 8 of [4]), sufficiency of the number of found invariants for forming a basis of $\mathrm{Inv}(\mathfrak{t}_0(n))$ is proved in the process of calculating them. Investigation of $\mathrm{Inv}(\mathfrak{t}_0(n))$ in this way gives us a sense of the specific features of the normalization procedure in the case of Lie algebras having nilradicals isomorphic (or closed) to $\mathfrak{t}_0(n)$.

For the algebra $\mathfrak{t}_0(n)$ we use a 'matrix' enumeration of the basis elements with an 'increasing' pair of indices, in a similar way to the canonical basis $\left\{E^n_{ij}, i < j\right\}$ of the isomorphic matrix algebra.

Hereafter E^n_{ij} (for fixed values of i and j) denotes the $n \times n$ matrix $(\delta_{ii'}\delta_{jj'})$ with i' and j' running the numbers of rows and column respectively, i.e., the $n \times n$ matrix with a unit element on the cross of the ith row and the jth column, and zero otherwise. $E^n = \mathrm{diag}(1, \dots, 1)$ is the $n \times n$ unity matrix. The indices i, j, k and l run at most from 1 to n. Only additional constraints on the indices are indicated.

Thus, the basis elements $e_{ij} \sim E^n_{ij}, i < j$, satisfy the commutation relations

$$[e_{ij}, e_{i'j'}] = \delta_{i'j} e_{ij'} - \delta_{ij'} e_{i'j},$$

where δ_{ij} is the Kronecker delta.

Let e^*_{ji}, x_{ji} and y_{ij} denote the basis element and the coordinate function in the dual space $\mathfrak{t}^*_0(n)$ and the coordinate function in $\mathfrak{t}_0(n)$, which correspond to the basis element $e_{ij}, i < j$. In particular, $\langle e^*_{j'i'}, e_{ij}\rangle = \delta_{ii'}\delta_{jj'}$. The reverse order of subscripts of the objects associated with the dual space $\mathfrak{t}^*_0(n)$ is justified by the simplification of a matrix representation of lifted invariants. We complete the sets of x_{ji} and y_{ij} in the matrices X and Y with zeros. Hence X is a strictly lower triangular matrix and Y is a strictly upper triangular one.

We reproduce lemma 1 from [4] together with its proof, since it is important for further consideration.

Lemma 1. *A complete set of independent lifted invariants of* $\mathrm{Ad}^*_{T_0(n)}$ *is exhaustively given by the expressions*

$$\mathcal{I}_{ij} = x_{ij} + \sum_{i<i'} b_{ii'} x_{i'j} + \sum_{j'<j} b_{j'j} x_{ij'} + \sum_{i<i',\, j'<j} b_{ii'} \widehat{b}_{j'j} x_{i'j'}, \qquad j < i,$$

where $B = (b_{ij})$ *is an arbitrary matrix from* $T_0(n)$, *and* $B^{-1} = (\widehat{b}_{ij})$ *is the inverse matrix of* B.

Proof. The adjoint action of $B \in T_0(n)$ on the matrix Y is $\mathrm{Ad}_B Y = BYB^{-1}$, i.e.,

$$\mathrm{Ad}_B \sum_{i<j} y_{ij} e_{ij} = \sum_{i<j} (BYB^{-1})_{ij} e_{ij} = \sum_{i\leqslant i'<j'\leqslant j} b_{ii'} y_{i'j'} \widehat{b}_{j'j} e_{ij}.$$

After changing $e_{ij} \to x_{ji}, y_{ij} \to e^*_{ji}, b_{ij} \leftrightarrow \widehat{b}_{ij}$ in the latter equality, we obtain the representation of the coadjoint action of B

$$\mathrm{Ad}^*_B \sum_{i<j} x_{ji} e^*_{ji} = \sum_{i\leqslant i'<j'\leqslant j} b_{j'j} x_{ji} \widehat{b}_{ii'} e^*_{j'i'} = \sum_{i'<j'} (BXB^{-1})_{j'i'} e^*_{j'i'}.$$

Therefore, the elements $\mathcal{I}_{ij}$, $j < i$, of the matrix $\mathcal{I} = BXB^{-1}$, $B \in T_0(n)$, form a complete set of the independent lifted invariants of $\mathrm{Ad}^*_{T_0(n)}$. □

Note 1. The centre of the group $T_0(n)$ is $Z(T_0(n)) = \{E^n + b_{1n}E^n_{1n}, b_{1n} \in \mathbb{F}\}$. The inner automorphism group of $\mathfrak{t}_0(n)$ is isomorphic to the factor-group $T_0(n)/Z(T_0(n))$ and hence its dimension is $\frac{1}{2}n(n-1) - 1$. The parameter b_{1n} in the above representation of the lifted invariants is not essential.

Below $A^{i_1,i_2}_{j_1,j_2}$, where $i_1 \leqslant i_2$, $j_1 \leqslant j_2$, denotes the submatrix $(a_{ij})^{i=i_1,\ldots,i_2}_{j=j_1,\ldots,j_2}$ of a matrix $A = (a_{ij})$. The conjugate value of k with respect to n is denoted by $\varkappa$, i.e. $\varkappa = n - k + 1$. The standard notation $|A| = \det A$ is used.

Theorem 1. *A basis of* $\mathrm{Inv}\big(\mathrm{Ad}^*_{T_0(n)}\big)$ *consists of the polynomials*

$$\big|X^{\varkappa,n}_{1,k}\big|, \qquad k = 1, \ldots, \left[\frac{n}{2}\right].$$

Proof. Under normalization we impose the following restriction on the lifted invariants $\mathcal{I}_{ij}$, $j < i$:

$$\mathcal{I}_{ij} = 0 \quad \text{if} \quad j < i, \quad (i, j) \neq (n - j' + 1, j'), \quad j' = 1, \ldots, \left[\frac{n}{2}\right].$$

It means that we do not only fix the values of the elements of the lifted invariant matrix $\mathcal{I}$, which are situated on the secondary diagonal, under the main diagonal. The other significant elements of $\mathcal{I}$ are given the value 0. As shown below, the chosen normalization is correct since it provides satisfying the conditions of proposition 1.

In view of the (triangular) structure of the matrices B and X the formula $\mathcal{I} = BXB^{-1}$, determining the lifted invariants implies that $BX = \mathcal{I}B$. This matrix equality is also significant for the matrix elements underlying the main diagonals of the left- and right-hand sides, i.e.,

$$x_{ij} + \sum_{i<i'} b_{ii'} x_{i'j} = \mathcal{I}_{ij} + \sum_{j'<j} \mathcal{I}_{ij'} b_{j'j}, \qquad j < i.$$

For convenience we divide the latter system under the chosen normalization conditions into four sets of subsystems

$$\begin{aligned}
S_1^k: &\quad x_{\varkappa j} + \sum_{i'>\varkappa} b_{\varkappa i'} x_{i'j} = 0, && i = \varkappa, \quad j < k, && k = 2, \ldots, \left[\frac{n}{2}\right],\\
S_2^k: &\quad x_{\varkappa k} + \sum_{i'>\varkappa} b_{\varkappa i'} x_{i'k} = \mathcal{I}_{\varkappa k}, && i = \varkappa, \quad j = k, && k = 1, \ldots, \left[\frac{n}{2}\right],\\
S_3^k: &\quad x_{\varkappa j} + \sum_{i'>\varkappa} b_{\varkappa i'} x_{i'j} = \mathcal{I}_{\varkappa k} b_{kj}, && i = \varkappa, \quad k < j < \varkappa, && k = 1, \ldots, \left[\frac{n}{2}\right] - 1,\\
S_4^k: &\quad x_{kj} + \sum_{i'>k} b_{ki'} x_{i'j} = 0, && i = k, \quad j < k, && k = 2, \ldots, \left[\frac{n+1}{2}\right],
\end{aligned}$$

and solve them one by one. The subsystem S_2^1 consists of the single equation $\mathcal{I}_{n1} = x_{n1}$ which gives the simplest form of the invariant corresponding to the centre of the algebra $\mathfrak{t}_0(n)$. For any fixed $k \in \{2, \ldots, [n/2]\}$ the subsystem $S_1^k \cup S_2^k$ is a well-defined system of linear equations with respect to $b_{\varkappa i'}$, $i' > \varkappa$, and $\mathcal{I}_{\varkappa k}$. Solving it, e.g., by the Cramer method, we obtain that $b_{\varkappa i'}$, $i' > \varkappa$, are expressions of $x_{i'j}$, $i' > \varkappa$, $j < k$, the explicit form of which is not essential in what follows, and

$$\mathcal{I}_{\varkappa k} = (-1)^{k+1} \frac{\big|X^{\varkappa,n}_{1,k}\big|}{\big|X^{\varkappa+1,n}_{1,k-1}\big|}, \qquad k = 2, \ldots, \left[\frac{n}{2}\right].$$

The combination of the found values of $\mathcal{I}_{\varkappa k}$ results in the invariants from the statement of the theorem. The functional independence of these invariants is obvious.

After substituting the expressions of $\mathcal{I}_{\varkappa k}$ and $b_{\varkappa i'}$, $i' > \varkappa$, via x's, into S_3^k, we trivially resolve S_3^k with respect to b_{kj} as an uncoupled system of linear equations. In performing the subsequent substitution of the calculated expressions for b_{kj} to S_4^k, for any fixed k, we obtain a well-defined system of linear equations, e.g., with respect to $b_{ki'}$, $i' > \varkappa$.

Under the normalization we express the non-normalized lifted invariants via x's and find a part of the parameters b's of the coadjoint action via x's and the other b's. No equations involving only x's are obtained. In view of proposition 1, this implies that the choice of the normalization constraints is correct and, therefore, the number of functionally independent invariants found is maximal, i.e., they form a basis of $\mathrm{Inv}\big(\mathrm{Ad}^*_{T_0(n)}\big)$. □

Corollary 1. *A basis of* $\mathrm{Inv}(\mathfrak{t}_0(n))$ *is formed by the Casimir operators*

$$\det(e_{ij})^{i=1,\ldots,k}_{j=n-k+1,\ldots,n}, \qquad k=1,\ldots,\left[\frac{n}{2}\right].$$

Proof. Since the basis elements corresponding to the coordinate functions from the constructed basis of $\mathrm{Inv}\big(\mathrm{Ad}^*_{T_0(n)}\big)$ commute, the symmetrization procedure is trivial. □

Note 2. The set of the invariants from theorem 1 can be easily found from the equality $\mathcal{I} = BXB^{-1}$ by the following empiric trick used in lemma 2 from [4]. For any fixed $k \in \{1,\ldots,[n/2]\}$ we restrict the equality to the submatrix with the row range $\varkappa,\ldots,n$ and the column range $1,\ldots,k$: $\mathcal{I}^{\varkappa,n}_{1,k} = B^{\varkappa,n}_{\varkappa,n}X^{\varkappa,n}_{1,k}(B^{-1})^{1,k}_{1,k}$. Since $\big|B^{\varkappa,n}_{\varkappa,n}\big| = \big|(B^{-1})^{1,k}_{1,k}\big| = 1$, we obtain $\big|\mathcal{I}^{\varkappa,n}_{1,k}\big| = \big|X^{\varkappa,n}_{1,k}\big|$, i.e., $\big|X^{\varkappa,n}_{1,k}\big|$ is an invariant of $\mathrm{Ad}^*_{T_0(n)}$ in view of the definition of an invariant. Functional independence of the constructed invariants is obvious. The proof of $N_{\mathfrak{t}_0(n)} = [n/2]$ is much more difficult (see lemma 3 of [4]).

4. Solvable algebra of upper triangular matrices

In a way analogous to the previous section, consider the solvable Lie algebra $\mathfrak{t}(n)$ isomorphic to one of the upper triangular $n\times n$ matrices. $\mathfrak{t}(n)$ has dimension $n(n+1)/2$. It is the Lie algebra of the Lie group $T(n)$ of nonsingular upper triangular $n\times n$ matrices.

Its basis elements are convenient to enumerate with a 'non-decreasing' pair of indices in a similar way to the canonical basis $\big\{E^n_{ij}, i\leqslant j\big\}$ of the isomorphic matrix algebra. Thus, the basis elements $e_{ij}\sim E^n_{ij}$, $i\leqslant j$, satisfy the commutation relations

$$[e_{ij}, e_{i'j'}] = \delta_{i'j}e_{ij'} - \delta_{ij'}e_{i'j},$$

where δ_{ij} is the Kronecker delta.

Hereafter the indices i, j, k and l again run at most from 1 to n. Only additional constraints on the indices are indicated.

The centre of $\mathfrak{t}(n)$ is one-dimensional and coincides with the linear span of the sum $e_{11}+\cdots+e_{nn}$ corresponding to the unity matrix E^n. The elements e_{ij}, $i<j$, and $e_{11}+\cdots+e_{nn}$ form a basis of the nilradical of $\mathfrak{t}(n)$, which is isomorphic to $\mathfrak{t}_0(n)\oplus\mathfrak{a}$. Here $\mathfrak{a}$ is the one-dimensional (Abelian) Lie algebra.

Let e^*_{ji}, x_{ji} and y_{ij} denote the basis element and the coordinate function in the dual space $\mathfrak{t}^*(n)$ and the coordinate function in $\mathfrak{t}(n)$, which correspond to the basis element e_{ij}, $i\leqslant j$. Thus, $\langle e^*_{j'i'}, e_{ij}\rangle = \delta_{ii'}\delta_{jj'}$. We complete the sets of x_{ji} and y_{ij} in the matrices X and Y with zeros. Hence X is a lower triangular matrix and Y is an upper triangular one.

Lemma 2. *A fundamental lifted invariant of* $\mathrm{Ad}^*_{T(n)}$ *is formed by the expressions*

$$\mathcal{I}_{ij} = \sum_{i\leqslant i',\, j'\leqslant j} b_{ii'}\widehat{b}_{j'j}x_{i'j'}, \qquad j\leqslant i,$$

where $B=(b_{ij})$ *is an arbitrary matrix from* $T(n)$, *and* $B^{-1}=(\widehat{b}_{ij})$ *is the inverse matrix of* B.

Proof. The adjoint action of $B \in T(n)$ on the matrix Y is $\mathrm{Ad}_B Y = BYB^{-1}$, i.e.

$$\mathrm{Ad}_B \sum_{i \leqslant j} y_{ij} e_{ij} = \sum_{i \leqslant j} (BYB^{-1})_{ij} e_{ij} = \sum_{i \leqslant i' \leqslant j' \leqslant j} b_{ii'} y_{i'j'} \widehat{b}_{j'j} e_{ij}.$$

After changing $e_{ij} \to x_{ji}, y_{ij} \to e^*_{ji}, b_{ij} \leftrightarrow \widehat{b}_{ij}$ in the latter equality, we obtain the representation for the coadjoint action of B

$$\mathrm{Ad}^*_B \sum_{i \leqslant j} x_{ji} e^*_{ji} = \sum_{i \leqslant i' \leqslant j' \leqslant j} b_{j'j} x_{ji} \widehat{b}_{ii'} e^*_{j'i'} = \sum_{i' \leqslant j'} (BXB^{-1})_{j'i'} e^*_{j'i'}.$$

Therefore, the elements $\mathcal{I}_{ij}$, $j \leqslant i$, of the matrix

$$\mathcal{I} = BXB^{-1}, \qquad B \in T(n),$$

form a complete set of the independent lifted invariants of $\mathrm{Ad}^*_{T(n)}$. □

Note 3. The centre of the group $T(n)$ is $Z(T(n)) = \{\beta E^n \mid \beta \in \mathbb{F}/\{0\}\}$. If $\mathbb{F} = \mathbb{C}$ then the group $T(n)$ is connected. In the real case the connected component $T_+(n)$ of the unity in $T(n)$ is formed by the matrices from $T(n)$ with positive diagonal elements, i.e., $T_+(n) \simeq T(n)\big/\mathbb{Z}_2^n$, where $\mathbb{Z}_2^n = \{\mathrm{diag}(\varepsilon_1, \ldots, \varepsilon_n) \mid \varepsilon_i = \pm 1\}$. The inner automorphism group $\mathrm{Int}(\mathfrak{t}(n))$ of $\mathfrak{t}(n)$ is isomorphic to the factor-group $T(n)/Z(T(n))$ (or $T_+(n)/Z(T(n))$ if $\mathbb{F}$ is real) and hence its dimension is $\frac{1}{2}n(n+1) - 1$. The value of one from the diagonal elements of the matrix B or a homogenous combination of them in the above representation of lifted invariants can be assumed inessential. It is evident from the proof of theorem 2 that in all cases, the invariant sets of the coadjoint representations of $\mathrm{Int}(\mathfrak{t}(n))$ and $\mathfrak{t}(n)$ coincide.

Let us remind that $A^{i_1,i_2}_{j_1,j_2}$, where $i_1 \leqslant i_2$, $j_1 \leqslant j_2$, denotes the submatrix $(a_{ij})^{i=i_1,\ldots,i_2}_{j=j_1,\ldots,j_2}$ of a matrix $A = (a_{ij})$. The conjugate value of k with respect to n is denoted by $\varkappa$, i.e. $\varkappa = n-k+1$.

Under the proof of the below theorem the following technical lemma on matrices is used.

Lemma 3. *Suppose* $1 < k < n$. *If* $|X^{\varkappa+1,n}_{1,k-1}| \neq 0$ *then for any* $\beta \in \mathbb{F}$

$$\left(\beta - X^{i,i}_{1,k-1}\left(X^{\varkappa+1,n}_{1,k-1}\right)^{-1} X^{\varkappa+1,n}_{j,j}\right) = \frac{(-1)^{k+1}}{\left|X^{\varkappa+1,n}_{1,k-1}\right|} \begin{vmatrix} X^{i,i}_{1,k-1} & \beta \\ X^{\varkappa+1,n}_{1,k-1} & X^{\varkappa+1,n}_{j,j} \end{vmatrix}.$$

In particular, $\left(x_{\varkappa k} - X^{\varkappa,\varkappa}_{1,k-1}\left(X^{\varkappa+1,n}_{1,k-1}\right)^{-1} X^{\varkappa+1,n}_{k,k}\right) = (-1)^{k+1}\left|X^{\varkappa+1,n}_{1,k-1}\right|^{-1}\left|X^{\varkappa,n}_{1,k}\right|$. *Analogously*

$$\left(x_{\varkappa j} - X^{\varkappa,\varkappa}_{1,k-1}\left(X^{\varkappa+1,n}_{1,k-1}\right)^{-1} X^{\varkappa+1,n}_{j,j}\right)\left(x_{jk} - X^{j,j}_{1,k-1}\left(X^{\varkappa+1,n}_{1,k-1}\right)^{-1} X^{\varkappa+1,n}_{k,k}\right)$$

$$= \frac{1}{\left|X^{\varkappa+1,n}_{1,k-1}\right|} \begin{vmatrix} X^{j,j}_{1,k} & \beta \\ X^{\varkappa,n}_{1,k} & X^{\varkappa,n}_{j,j} \end{vmatrix} + \frac{\left|X^{\varkappa,n}_{1,k}\right|}{\left|X^{\varkappa+1,n}_{1,k-1}\right|^2} \begin{vmatrix} X^{j,j}_{1,k-1} & \beta \\ X^{\varkappa+1,n}_{1,k-1} & X^{\varkappa+1,n}_{j,j} \end{vmatrix}.$$

Theorem 2. *A basis of* $\mathrm{Inv}(\mathrm{Ad}^*_{T(n)})$ *is formed by the rational expressions*

$$\frac{1}{\left|X^{\varkappa,n}_{1,k}\right|} \sum_{j=k+1}^{\varkappa-1} \begin{vmatrix} X^{j,j}_{1,k} & x_{jj} \\ X^{\varkappa,n}_{1,k} & X^{\varkappa,n}_{j,j} \end{vmatrix}, \qquad k = 0, \ldots, \left[\frac{n-1}{2}\right],$$

where $\left|X^{n+1,n}_{1,0}\right| := 1$.

Proof. We choose the following normalization restriction on the lifted invariants $\mathcal{I}_{ij}$, $j \leqslant i$:

$$\mathcal{I}_{n-j+1,j} = 1, \quad j = 1, \ldots, \left[\frac{n}{2}\right],$$

$$\mathcal{I}_{ij} = 0 \quad \text{if} \quad j \leqslant i, \quad (i,j) \neq (j',j'), \quad (n-j'+1, j'), \quad j' = 1, \ldots, \left[\frac{n+1}{2}\right].$$

This means that we do not only fix the values of the elements of the lifted invariant matrix $\mathcal{I}$, which are situated on the main diagonal over or on the secondary diagonal. The elements of the secondary diagonal underlying the main diagonal are given a value of 1. The other significant elements of $\mathcal{I}$ are given a value 0. As shown below, the imposed normalization provides satisfying the conditions of proposition 1 and, therefore, is correct.

Similarly to the case of strictly triangular matrices, in view of the (triangular) structure of the matrices B and X the formula $\mathcal{I} = BXB^{-1}$ determining the lifted invariants implies that $BX = \mathcal{I}B$. This matrix equality is significant for the matrix elements lying not over the main diagonals of the left and right hand sides, i.e.,

$$\sum_{i\leqslant i'} b_{ii'}x_{i'j} = \sum_{j'\leqslant j} \mathcal{I}_{ij'}b_{j'j}, \qquad j\leqslant i.$$

For convenience we again divide the latter system under the chosen normalization conditions into four sets of subsystems

$$\begin{aligned}
&S_1^k: && \sum_{i'\geqslant\varkappa} b_{\varkappa i'}x_{i'j} = 0, && i=\varkappa, \quad j<k, && k=2,\ldots,\left[\frac{n}{2}\right],\\
&S_2^k: && \sum_{i'\geqslant\varkappa} b_{\varkappa i'}x_{i'j} = b_{kj}, && i=\varkappa, \quad k\leqslant j\leqslant\varkappa, && k=1,\ldots,\left[\frac{n}{2}\right],\\
&S_3^k: && \sum_{i'\geqslant k} b_{ki'}x_{i'j} = 0, && i=k, \quad j<k, && k=2,\ldots,\left[\frac{n+1}{2}\right],\\
&S_4^k: && \sum_{i'\geqslant k} b_{ki'}x_{i'k} = b_{kk}\mathcal{I}_{kk}, && i=k, \quad j<k, && k=1,\ldots,\left[\frac{n+1}{2}\right],
\end{aligned}$$

and solve them one by one. The subsystem S_2^1 consists of the equations

$$b_{1j} = b_{nn}x_{nj}$$

which are already solved with respect to b_{1j}. For any fixed $k\in\{2,\ldots,[n/2]\}$ the subsystem $S_1^k \cup S_2^k$ is a well-defined system of linear equations with respect to $b_{\varkappa i'}, i'>\varkappa$, and $b_{kj}, k\leqslant j\leqslant\varkappa$. We can solve the subsystem S_1^k with respect to $b_{\varkappa i'}, i'>\varkappa$:

$$B^{\varkappa,\varkappa}_{\varkappa+1,n} = -b_{\varkappa\varkappa}X^{\varkappa,\varkappa}_{1,k-1}\left(X^{\varkappa+1,n}_{1,k-1}\right)^{-1},$$

and then substitute the obtained values into the subsystem S_2^k. Another way is to find the expressions for $b_{kj}, k\leqslant j\leqslant\varkappa$, by the Cramer method, from the whole system $S_1^k\cup S_2^k$ at once since only these parameters are further considered. As a result, they have two representations via $b_{\varkappa\varkappa}$ and x's:

$$b_{kj} = b_{\varkappa\varkappa}\left(x_{\varkappa j} - X^{\varkappa,\varkappa}_{1,k-1}\left(X^{\varkappa+1,n}_{1,k-1}\right)^{-1}X^{\varkappa+1,n}_{j,j}\right) = \frac{(-1)^{k+1}b_{\varkappa\varkappa}}{\left|X^{\varkappa+1,n}_{1,k-1}\right|}\begin{vmatrix} X^{\varkappa,\varkappa}_{1,k-1} & x_{\varkappa j}\\ X^{\varkappa+1,n}_{1,k-1} & X^{\varkappa+1,n}_{j,j}\end{vmatrix},$$

where $k\leqslant j\leqslant\varkappa$. In particular,

$$b_{kk} = (-1)^{k+1}b_{\varkappa\varkappa}\left|X^{\varkappa+1,n}_{1,k-1}\right|^{-1}\left|X^{\varkappa,n}_{1,k}\right|.$$

Analogously, for any fixed $k\in\{2,\ldots,[(n+1)/2]\}$ the subsystem S_3^k is a well-defined system of linear equations with respect to $b_{kj}, j>\varkappa$, and it implies

$$B^{k,k}_{\varkappa+1,n} = -\sum_{k\leqslant j\leqslant\varkappa} b_{kj}X^{j,j}_{1,k-1}\left(X^{\varkappa+1,n}_{1,k-1}\right)^{-1}.$$

Substituting the found expressions for b's into the equations of the subsystems S_4^k, we completely exclude the parameters b's and obtain expressions of $\mathcal{I}_{kk}$ only via x's. Thus, under $k = 1$

$$\mathcal{I}_{11} = \frac{1}{b_{11}}\sum_i b_{1i}x_{i1} = \frac{b_{nn}}{b_{11}}\sum_i x_{ni}x_{i1} = \frac{1}{x_{n1}}\sum_i x_{ni}x_{i1} = \frac{1}{x_{n1}}\sum_i \begin{vmatrix} x_{i1} & x_{ii} \\ x_{n1} & x_{ni}\end{vmatrix} + \sum_i x_{ii},$$

where the summation range in the first sum can be bounded by 2 and $n-1$ since for $i = 1$ and $i = n$ the determinants are equal to 0. In the case $k \in \{2, \ldots, [(n+1)/2]\}$

$$b_{kk}\mathcal{I}_{kk} = \sum_{k\leqslant i} b_{ki}x_{ik} = \sum_{k\leqslant j\leqslant \varkappa} b_{kj}x_{jk} + \sum_{\varkappa<i} b_{ki}x_{ik} = \sum_{k\leqslant i\leqslant\varkappa} b_{kj}\left(x_{jk} - X^{j,j}_{1,k-1}\left(X^{\varkappa+1,n}_{1,k-1}\right)^{-1}X^{\varkappa+1,n}_{k,k}\right)$$

$$= b_{\varkappa\varkappa}\sum_{k\leqslant i\leqslant\varkappa}\left(x_{\varkappa j} - X^{\varkappa,\varkappa}_{1,k-1}\left(X^{\varkappa+1,n}_{1,k-1}\right)^{-1}X^{\varkappa+1,n}_{j,j}\right)\left(x_{jk} - X^{j,j}_{1,k-1}\left(X^{\varkappa+1,n}_{1,k-1}\right)^{-1}X^{\varkappa+1,n}_{k,k}\right).$$

After using the representation for b_{nn} and manipulations with submatrices of X (see lemma 3), we derive that

$$\mathcal{I}_{kk} = \frac{(-1)^{k+1}}{\left|X^{\varkappa,n}_{1,k}\right|}\sum_{k\leqslant i\leqslant\varkappa}\begin{vmatrix} X^{i,i}_{1,k} & x_{ii} \\ X^{\varkappa,n}_{1,k} & X^{\varkappa,n}_{i,i}\end{vmatrix} + \frac{(-1)^{k+1}}{\left|X^{\varkappa+1,n}_{1,k-1}\right|}\sum_{k\leqslant i\leqslant\varkappa}\begin{vmatrix} X^{i,i}_{1,k-1} & x_{ii} \\ X^{\varkappa+1,n}_{1,k-1} & X^{\varkappa+1,n}_{i,i}\end{vmatrix},$$

where $k = 2, \ldots, [(n+1)/2]$. The summation range in the first sum can be taken from $k+1$ and $\varkappa - 1$ since for $i = k$ and $i = \varkappa$ the determinants are equal to 0.

The combination of the found values of $\mathcal{I}_{kk}$ in the following way

$$\tilde{\mathcal{I}}_{00} = \sum_{j=1}^{[\frac{n+1}{2}]}\mathcal{I}_{jj} = \sum_i x_{ii}, \quad \tilde{\mathcal{I}}_{kk} = (-1)^{k+1}\mathcal{I}_{kk} - \tilde{\mathcal{I}}_{k-1,k-1}, \quad k = 1, \ldots, \left[\frac{n-1}{2}\right],$$

results in the invariants $\tilde{\mathcal{I}}_{k'k'}$, $k' = 0, \ldots, [(n-1)/2]$, from the statement of the theorem. The functional independence of these invariants is obvious.

Under the normalization we express the non-normalized lifted invariants via x's and find a part of the parameters b's of the coadjoint action via x's and the other b's. No equations involving only x's are obtained. In view of proposition 1, this implies that the choice of the normalization constraints is correct, i.e., the number of the found functionally independent invariant is maximal and, therefore, they form a basis of $\mathrm{Inv}(\mathrm{Ad}^*_{T(n)})$. □

Note 4. An expanded form of the invariants from theorem 2 is

$$\sum_{j=1}^{n} x_{jj}, \qquad \frac{\sum_{j=2}^{n-1}\begin{vmatrix} x_{j1} & x_{jj} \\ x_{n1} & x_{nj}\end{vmatrix}}{x_{n1}}, \qquad \frac{\sum_{j=3}^{n-2}\begin{vmatrix} x_{j1} & x_{j2} & x_{jj} \\ x_{n-1,1} & x_{n-1,2} & x_{n-1,j} \\ x_{n1} & x_{n2} & x_{nj}\end{vmatrix}}{\begin{vmatrix} x_{n-1,1} & x_{n-1,2} \\ x_{n1} & x_{n2}\end{vmatrix}}, \quad \ldots.$$

The first invariant corresponds to the centre of $\mathfrak{t}(n)$. The invariant tuple ends with

$$\frac{\left|X^{\frac{n-1}{2},n}_{1,\frac{n+1}{2}}\right|}{\left|X^{\frac{n+1}{2},n}_{1,\frac{n-1}{2}}\right|} \quad \text{if } n \text{ is odd} \qquad \text{and} \qquad \frac{\sum_{j=\frac{n}{2}}^{\frac{n}{2}+1}\begin{vmatrix} X^{j,j}_{1,\frac{n}{2}-1} & x_{jj} \\ X^{\frac{n}{2}+2,n}_{1,\frac{n}{2}-1} & X^{\frac{n}{2}+2,n}_{j,j}\end{vmatrix}}{\left|X^{\frac{n}{2}+2,n}_{1,\frac{n}{2}-1}\right|} \quad \text{if } n \text{ is even}.$$

Corollary 2. *A basis of* $\mathrm{Inv}(\mathfrak{t}(n))$ *consists of the rational invariants*

$$\hat{\mathcal{I}}_k = \frac{1}{\left|\mathcal{E}^{1,k}_{\varkappa,n}\right|} \sum_{j=k+1}^{n-k} \begin{vmatrix} \mathcal{E}^{1,k}_{j,j} & \mathcal{E}^{1,k}_{\varkappa,n} \\ e_{jj} & \mathcal{E}^{j,j}_{\varkappa,n} \end{vmatrix}, \qquad k = 0, \dots, \left[\frac{n-1}{2}\right],$$

where $\mathcal{E}^{i_1,i_2}_{j_1,j_2}$, $i_1 \leqslant i_2$, $j_1 \leqslant j_2$, *denotes the matrix* $(e_{ij})^{i=i_1,\dots,i_2}_{j=j_1,\dots,j_2}$, $\left|\mathcal{E}^{1,0}_{n+1,n}\right| := 1$, $\varkappa = n - k + 1$.

Proof. By expanding the determinants in any elements of the basis from theorem 2, we obtain a rational expression for x's. Each monomial in the numerator or the denominator contains coordinate functions such that corresponding basis elements commute. Therefore, the symmetrization procedure is trivial. Since $x_{ij} \sim e_{ji}$, $j < i$, it is necessary to transpose the matrices in the obtained expressions of invariants, in order to improve the representation. □

Note 5. The invariants from corollary 2 can be rewritten as

$$\hat{\mathcal{I}}_k = \frac{1}{\left|\mathcal{E}^{1,k}_{\varkappa,n}\right|} \sum_{j=k+1}^{n-k} \begin{vmatrix} \mathcal{E}^{1,k}_{j,j} & \mathcal{E}^{1,k}_{\varkappa,n} \\ 0 & \mathcal{E}^{j,j}_{\varkappa,n} \end{vmatrix} + (-1)^{k+1} \sum_{j=k+1}^{n-k} e_{jj}, \qquad k = 0, \dots, \left[\frac{n-1}{2}\right].$$

In particular, $\hat{\mathcal{I}}_0 = \sum_j e_{jj}$.

5. Solvable algebra of special upper triangular matrices

The Lie algebra $\mathfrak{st}(n)$ of the special (i.e., having zero traces) upper triangular $n \times n$ matrices is imbedded in a natural way in $\mathfrak{t}(n)$ as an ideal. $\dim \mathfrak{st}(n) = \frac{1}{2}n(n+1) - 1$. Moreover,

$$\mathfrak{t}(n) = \mathfrak{st}(n) \oplus Z(\mathfrak{t}(n)),$$

where $Z(\mathfrak{t}(n)) = \langle e_{11} + \dots + e_{nn} \rangle$ is the centre of $\mathfrak{t}(n)$, which corresponds to the one-dimensional Abelian Lie algebra of the matrices proportional to E^n. Due to this fact we can construct a basis of $\mathrm{Inv}(\mathfrak{st}(n))$ without the usual calculations involved in finding the basis of $\mathrm{Inv}(\mathfrak{t}(n))$. It is well known that if the Lie algebra $\mathfrak{g}$ is decomposable into the direct sum of Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ then the union of bases of $\mathrm{Inv}(\mathfrak{g}_1)$ and $\mathrm{Inv}(\mathfrak{g}_2)$ is a basis of $\mathrm{Inv}(\mathfrak{g})$. A basis of $\mathrm{Inv}(Z(\mathfrak{t}(n)))$ obviously consists of only one element, e.g., $e_{11} + \dots + e_{nn}$. Therefore, the cardinality of the basis of $\mathrm{Inv}(\mathfrak{st}(n))$ is equal to the cardinality of the basis of $\mathrm{Inv}(\mathfrak{t}(n))$ minus 1, i.e., $[(n-1)/2]$. To construct a basis of $\mathrm{Inv}(\mathfrak{st}(n))$, it is enough for us to rewrite $[(n-1)/2]$ functionally independent combinations of elements from a basis of $\mathrm{Inv}(\mathfrak{t}(n))$ via elements of $\mathfrak{st}(n)$ and to exclude the central element from the basis.

The following basis in $\mathfrak{st}(n)$ is chosen as a subalgebra of $\mathfrak{t}(n)$:

$$e_{ij}, \qquad i < j, \qquad f_k = \frac{n-k}{n} \sum_{i=1}^{k} e_{ii} - \frac{k}{n} \sum_{i=k+1}^{n} e_{ii}, \quad k = 1, \dots, n-1.$$

(Usage of this basis allows for the presentation of our results in such a form that their identity with proposition 1 from [23] becomes absolutely evident.) The commutation relations of $\mathfrak{st}(n)$ in the chosen basis are

$$\begin{aligned}
&[e_{ij}, e_{i'j'}] = \delta_{i'j} e_{ij'} - \delta_{ij'} e_{i'j}, && i < j, \quad i' < j'; \\
&[f_k, f_{k'}] = 0, && k, k' = 1, \dots, n-1; \\
&[f_k, e_{ij}] = 0, && i < j \leqslant k \quad \text{or} \quad k \leqslant i < j; \\
&[f_k, e_{ij}] = e_{ij}, && i \leqslant k \leqslant j, \quad i < j
\end{aligned}$$

and, therefore, coincide with those of the algebra $L(n, n-1)$ from [22], i.e., $L(n, n-1)$ is isomorphic to $\mathfrak{st}(n)$. Combining this observation with lemma 6 of [22] results in the following theorem.

Theorem 3. *The Lie algebra* $\mathfrak{st}(n)$ *has the maximal number of dimensions (equal to* $\frac{1}{2}n(n+1)-1$*) among the solvable Lie algebras which have nilradicals isomorphic to* $\mathfrak{t}_0(n)$*. It is the unique algebra with such a property.*

Theorem 4. *A basis of* $\mathrm{Inv}(\mathfrak{st}(n))$ *consists of the rational invariants*

$$\check{\mathcal{I}}_k = \frac{(-1)^{k+1}}{\left|\mathcal{E}^{1,k}_{\varkappa,n}\right|}\sum_{j=k+1}^{n-k}\begin{vmatrix}\mathcal{E}^{1,k}_{j,j} & \mathcal{E}^{1,k}_{\varkappa,n}\\ 0 & \mathcal{E}^{j,j}_{\varkappa,n}\end{vmatrix} + f_k - f_{n-k}, \qquad k=1,\ldots,\left[\frac{n-1}{2}\right],$$

where $\mathcal{E}^{i_1,i_2}_{j_1,j_2}$, $i_1\leqslant i_2$, $j_1\leqslant j_2$, *denotes the matrix* $(e_{ij})^{i=i_1,\ldots,i_2}_{j=j_1,\ldots,j_2}$, $\left|\mathcal{E}^{1,0}_{n+1,n}\right| := 1$, $\varkappa = n-k+1$.

Proof. It is enough to observe (see note 5) that

$$\check{\mathcal{I}}_k = (-1)^{k+1}\hat{\mathcal{I}}_k + \frac{n-2k}{n}\hat{\mathcal{I}}_0, \qquad k=1,\ldots,\left[\frac{n-1}{2}\right].$$

These combinations of elements from a basis of $\mathrm{Inv}(\mathfrak{t}(n))$ are functionally independent. They are expressed via elements of $\mathfrak{st}(n)$. Their number is $[(n-1)/2]$. Therefore, they form a basis of $\mathrm{Inv}(\mathfrak{st}(n))$. □

6. Conclusion and discussion

In this paper, we extend our purely algebraic approach for computing invariants of Lie algebras by means of moving frames [3, 4] to the classes of Lie algebras $\mathfrak{t}_0(n)$, $\mathfrak{t}(n)$ and $\mathfrak{st}(n)$ of strictly, non-strictly and special upper triangular matrices of an arbitrary fixed dimension n. In contrast to the conventional infinitesimal method which involves solving an associated system of PDEs, the main steps of the applied algorithm are the construction of the matrix $B(\theta)$ of inner automorphisms of the Lie algebra under consideration, and the exclusion of the parameters θ from the algebraic system $\mathcal{I} = \check{x}\cdot B(\theta)$ in some way. The version of the algorithm, applied in this paper, is distinguished in that a special usage of the normalization procedure when the number, and a form of elements in a functional basis of an invariant set, are determined by excluding the parameters simultaneously.

A basis of $\mathrm{Inv}(\mathfrak{t}_0(n))$ was already known and constructed by both the infinitesimal method [23] and the algebraic algorithm with an elegant but empiric technique of excluding the parameters [4]. Note that the proof introduced in [23] is very sophisticated and was completed only due to the thorough mastery of the used infinitesimal method. A form of elements from a functional basis of $\mathrm{Inv}(\mathfrak{t}_0(n))$ was guessed via calculation of bases for a number of small n's and then justified with the infinitesimal method, and both the testing steps (on invariance and on sufficiency of number) were quite complicated.

Invariants of $\mathfrak{t}_0(n)$ are considered in this paper in order to demonstrate the advantages of the normalization technique and to pave the way for further applications of this technique to the more complicated algebras $\mathfrak{t}(n)$ and $\mathfrak{st}(n)$, being too complex for the infinitesimal method (only the lowest few were completely investigated there). First the invariants of the algebras $\mathfrak{t}(n)$ and $\mathfrak{st}(n)$ are exhaustively studied in this paper. The performed calculations are simple and clear since the normalization procedure is reduced by the choice of natural coordinates on the inner automorphism groups and by the use of a special normalization technique to solving a *linear* system of algebraic equations. The results obtained for $\mathrm{Inv}(\mathfrak{st}(n))$ in theorem 4 completely agree with the conjecture formulated as proposition 1 in [23] on the number and form of basis elements of this invariant set.

A direct extension of the present investigation is to describe the invariants of the subalgebras of $\mathfrak{st}(n)$, which contain $\mathfrak{t}_0(n)$. Such subalgebras exhaust the set of solvable

Lie algebras which can be embedded in the matrix Lie algebra $\mathfrak{gl}(n)$ and have the nilradicals isomorphic to $\mathfrak{t}_0(n)$. A technique similar to that used in this paper can be applied. The main difficulties will be created by breaking in symmetry and complication of coadjoint representations. The question on ways of investigation of the other solvable Lie algebras with the nilradicals isomorphic to $\mathfrak{t}_0(n)$ remains open. (See, e.g., [22] for classification of the algebras of such type.)

A more general problem is to circumscribe an applicability domain of the developed algebraic method. It has been already applied only to the low-dimensional Lie algebras and a wide range of classes of solvable Lie algebras in [3, 4] and this paper. The next step which should be performed is the extension of the method to classes of unsolvable Lie algebras of arbitrary dimensions, e.g., with fixed structures of radicals or Levi factors. An adjoining problem is the implementation of the algorithm with symbolic calculation systems. Similar work has already begun in the framework of the general method of moving frames, e.g., in the case of rational invariants for rational actions of algebraic groups [11]. Some other possibilities on the applications of the algorithm are outlined in [4].

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